1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 5, Module 3, Rotational kinematics -Equation of Motion Chapter 7, System of particles and Rotational Motion
Module Id	Keph_10703_eContent
Pre-requisites	Kinematics, laws of motion, basic vector algebra, center of mass, translatory motion, circular motion
Objectives	After going through this lesson, the learners will be able to: • **Define** quantities for measuring rotational motion like angular displacement, angular velocity and angular acceleration • **Derive and apply** rotational equation of motion • **Compare** rotational motion and motion in straight line • **Understand** rotational kinematics
Keywords	Angular displacement, angular velocity, angular acceleration, direction of angular velocity, direction of angular acceleration

2. Development Team

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1. UNIT SYLLABUS

Unit V: Motion of System of Particles and Rigid body

Chapter 7: System of particles and Rotational Motion

Centre of mass of a two-particle system; momentum conservation and centre of mass motion. Centre of mass of a rigid body; Centre of mass of a uniform rod.

Moment of a force; torque; angular momentum; law of conservation of angular momentum and its applications.

Equilibrium of rigid bodies; rigid body rotation and equations of rotational motion; comparison of linear and rotational motions.

Moment of inertia; radius of gyration; values of moments of inertia for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 8 Modules

The above unit has been divided into 8 modules for better understanding.

Module 1	• Rigid body • Centre of mass • Distribution of mass • Types of motion: Translatory, circulatory and rotatory
Module 2	• Centre of mass • Application of centre of mass to describe motion • Motion of centre of mass
Module 3	• Analogy of circular motion of a point particle about a point and different points on a rigid body about an axis • Relation $v = r\omega$ • Kinematics of rotational motion • Equations for uniformly accelerated rotational motion
Module 4	• Moment of inertia • Difference between mass and moment of inertia • Derivation of value of moment of inertia for a lamina about a vertical axis perpendicular to the plane of the lamina • S I Unit • Radius of gyration • Perpendicular and Parallel axis theorems
Module 5	• Torque • Types of torque • Dynamics of rotational motion • $\tau = I\alpha$
Module 6	• Equilibrium of rigid bodies • Condition of net force and net torque • Applications
Module 7	• Law of conservation of angular momentum and its applications. • Applications
Module 8	• Rolling on plane surface • Horizontal • Inclined surface • Applications

Module 3

3. WORDS YOU MUST KNOW

- Rigid body: An object for which individual particles continue to be at the same separation over a period of time.
- Point object: If the position of an object changes by distances much larger than the dimensions of the body the body may be treated as a point object.
- Frame of reference: Any reference frame the coordinates (x, y, z), which indicate the change in position of object with time.
- Observer: Someone who is observing objects from any frame
- Rest: A body is said to be at rest if it does not change its position with surroundings with time
- Motion: A body is said to be in motion if it changes its position with respect to its surroundings with time
- Time elapsed: Time interval between any two observations of an object.
- Motion in one dimension: When the position of an object can be shown by change in any one coordinate out of the three (x, y, z), also called motion in a straight line.
- Motion in two dimension: When the position of an object can be shown by changes any two coordinate out of the three (x, y, z), also called motion in a plane.
- Motion in three dimension: When the position of an object can be shown by changes in all three coordinate out of the three (x, y, z).
- Distance: The length of the path an object has moved from its starting position to a final position in certain time.SI unit m , this can be zero or positive.
- Displacement: The distance an object has moved from its starting position to a final position in a particular direction SI unit: m, this can be zero , positive or negative .
- Position vector: A vector representing the location of a point in space with respect to a fixed frame of reference
- Force: A push or a pull that can change the state of rest or motion of a body. It can also deform a body.

- Center of mass: The centre of mass of a system of particles moves as if all the mass of the system was concentrated at this centre of mass and all external forces were applied at that point.
- Rectilinear motion: In rigid body motion along a straight path all point on the body move with the same speed and in the same direction.
- Curvilinear motion: In rigid body motion along a curved path where all points on the body travel in parallel curved tracks with the same speed.
- Circular motion: In a rigid body motion along a circular track the centre of mass of the body maintains a fixed distance from the center of the circular track

4. INTRODUCTION

In the first module of unit 5, we discussed that a body with a fixed size and shape is termed as rigid body and about possible motions it can show namely translation, rotational and general plane motion.

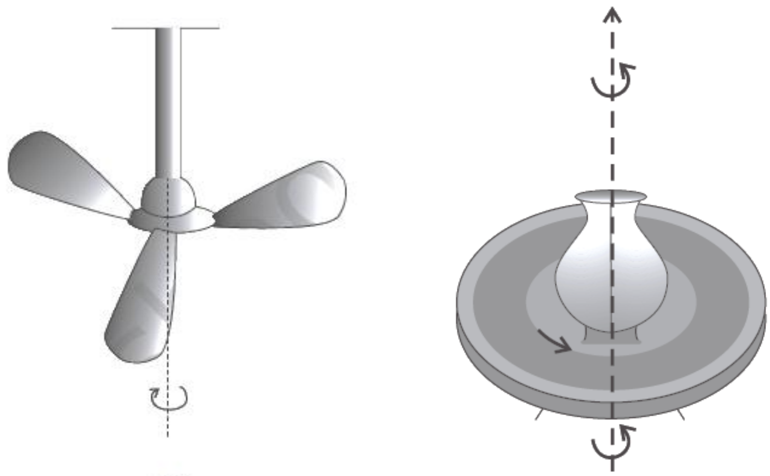
Though, we have been more concerned about their translational motion in the previous modules.

Now we will try to describe rotational motion and define quantities to measure the same. The concepts learnt here would be helpful in decoding and describing the general plane motion.

To start we will consider rotational motion of rigid bodies about "fixed axis". The axis of rotation of a rotating body is said to be fixed if its position is fixed with respect to the rotating body and direction is fixed with respect to an inertial frame; example, Ferris wheel or giant wheel in amusement park, overhead fan, door rotating about its hinges etc.

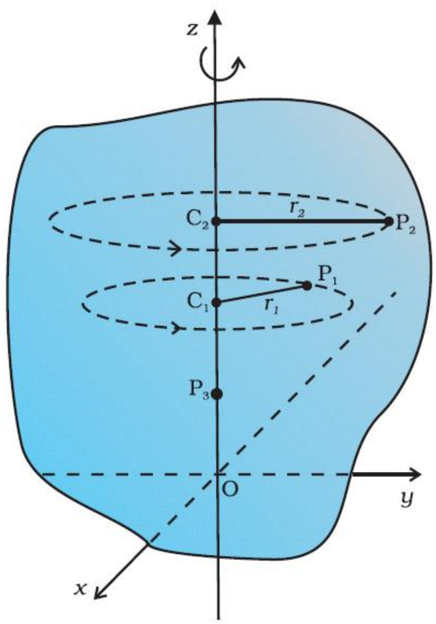
A rigid body is said to have **rotational motion** of about a fixed axis when the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis. Interesting to note if this axis, called the **axis of rotation intersects the rigid body**, **the particles located on the axis have zero velocity and zero acceleration.**

If you look around, you will come across many examples of rotation about an axis, a ceiling fan (shown in figure below), a potter's wheel, a giant wheel in a fair, a merry-go-round and so on.



Rotation about a fixed axis
(a) A ceiling fan (b) A potter's wheel

Let us try to understand what we mean by rotation and what are its characterizes!!



The figure above shows the rotational motion of a rigid body about a fixed axis (the z-axis of the frame of reference).

You may notice that in rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

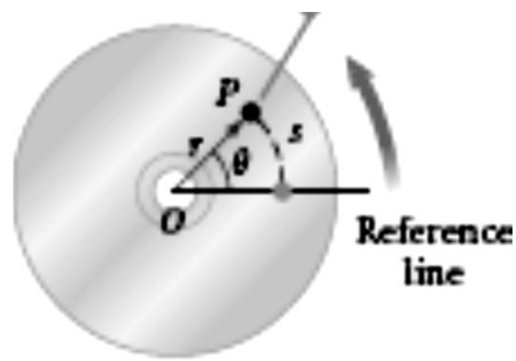
Let P_1 be a particle of the rigid body, arbitrarily chosen and at a distance r_1 from fixed axis. The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis. The circle lies in a plane perpendicular to the axis.

The figure also shows another particle P_2 of the rigid body, P_2 is at a distance r_2 from the fixed axis. The particle P_2 moves in a circle of radius r_2 and with centre C_2 on the axis. This circle, too, lies in a plane perpendicular to the axis.

Note that the circles described by P_1 and P_2 may lie in different planes; both these planes, however, are perpendicular to the fixed axis. For any particle on the axis like P_3, $r = 0$. All such particles, lying on the axis of rotation, remain stationary while the body rotates. This is expected since the axis is fixed.

We will try and give specific vocabulary in order to describe rotation

5. ANGULAR POSITION AND ANGULAR DISPLACEMENT



Consider a disc rotating about a fixed axis perpendicular to the plane of the figure and passing through the center of the disc at O.

This axis will be perpendicular to the plane containing the disc

A small element of the disc (as a particle at P) is at a fixed distance r from the origin and rotates about it in a circle of radius r. (In fact, every element of the disc undergoes circular motion about O.)

In this representation, the angle θ changes in time while r remains constant.

As the particle moves along the circle from the reference line, which is at angle $\theta = 0$, it moves through an arc of length s as in Figure.

The arc length s is related to the angle θ through the relationship

$$s = \theta r$$

{Angle subtended at the center of a circle by radius vector for one complete rotation = circumference /radius}

As the disc is a rigid object, the particle moves through an angle θ from the reference line, every other particles on the object rotates through the same angle θ.

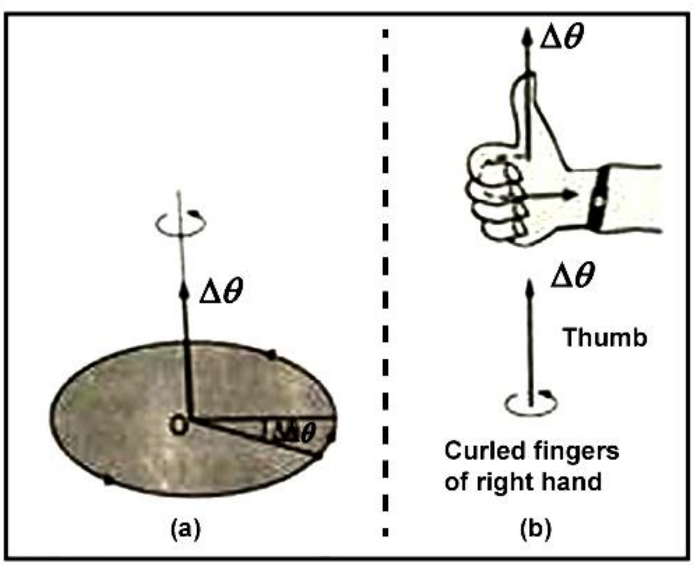
Therefore, we can associate the angle θ with the entire rigid object as well as with an individual particle, which allows us to define the angular position of a rigid object in its rotational motion.

We choose a reference line on the object, such as a line connecting O and a chosen particle on the object. The angular position of the rigid object is the angle θ between this reference line on the object and the fixed reference line in space, which is often chosen as the x axis. As the particle in question on our rigid object travels from position θ_i to position θ_f in a time interval Δt as in Figure, the reference line fixed to the object sweeps out an angle

$$\boldsymbol{\Delta\theta = \theta_f - \theta_i}$$

This quantity **Δθ is defined as the angular displacement of the rigid object**. It has both magnitude and direction.

The magnitude can be measured in terms of **radians (SI units)** and its direction can be specified in terms of x, y and z axes with the **right hand rule**.
According to it **when the fingers of the right hand curl in the sense of rotation, the thumb indicates the direction of the angular displacement.**



Angular displacement has both magnitude and direction;

It is interesting to find out whether it is a scalar or a vector quantity.

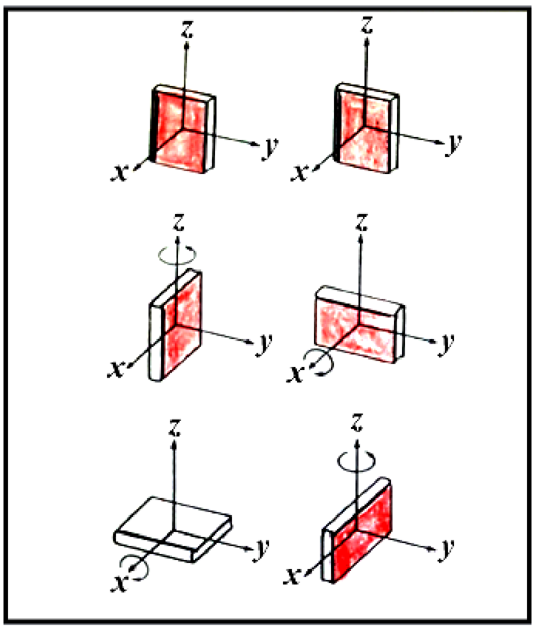
Let's do an activity with a book.

Consider a rotation of a book through a finite angle of 90° or π/2 only. Let it be rotated (θ_1) about z axis first followed by rotation (θ_2) about x axis, as shown in the figure. The book attains a certain final orientation as shown in figure.

Now if the same book starting from same initial orientations is made to rotate by 90° about x axis first and then followed by rotation of 9.0° about z axis, **the order of rotation is made reversed.**

One can observe that the final orientation of the book depends on the whether the rotation through $\theta_1$or θ_2 is performed first or one can even say the order of rotation can decide the final orientation of the book.

Therefore $$\theta_1 + \theta_2 \neq \theta_2 + \theta_1$$

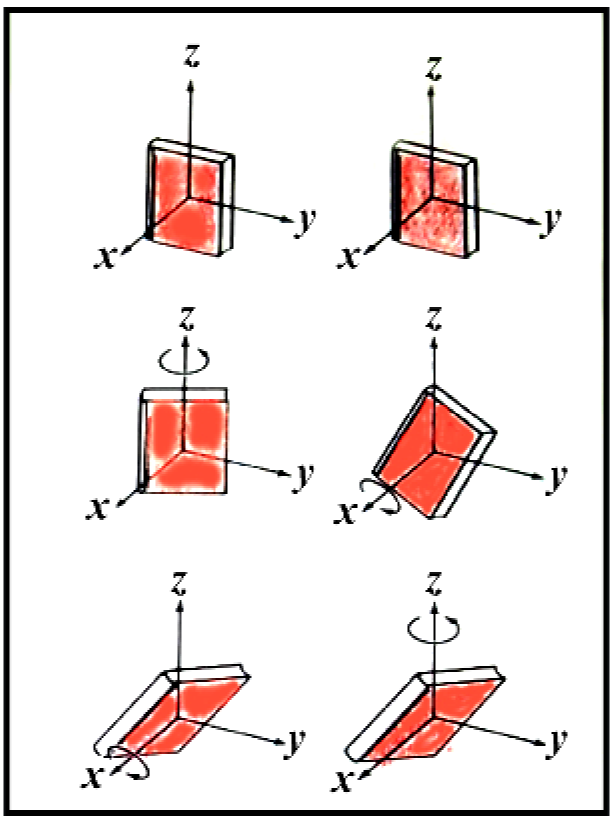


The book is rotated by 90^0 about each of the axis z and x. The final orientation depends upon the order in which rotation is performed.

The finite angular displacements do not obey the commutative property of the vector addition and therefore, finite angular displacements are not vectors.

It is **interesting if we consider small rotations** about the same axis, as done for finite angular displacements. Let it be rotated ($d\theta_1$) about z axis first followed by rotation ($d\theta_2$) about x axis, as shown in the figure below. The book attains a certain final orientation as shown in figure below. Now if the same book starting from same initial orientations is made to rotate about x axis first and then followed by rotation about z axis, the order of rotation is made reversed. One can observe that the final orientation of the book does not depends on the whether the rotation through $d\theta_1$ or $d\theta_2$ is performed first or one can even say the order of rotation does not decide the final orientation of the book.

Therefore $$d\theta_1 + d\theta_2 = d\theta_2 + d\theta_1$$



The book is rotated through small angle. The final orientation does not depend on the order of the rotation

Hence the infinitesimal rotations follow the laws of vector addition and therefore are vector quantities.

6. ANGULAR VELOCITY

The rate at which this angular displacement occurs can vary. If the rigid object spins rapidly, the displacement can occur in a short time interval, if it rotates slowly or the same displacement occurs in a longer time interval.

These different rotation rates can be quantified by defining the **average angular speed ω_{avg}** as the ratio of the angular displacement of a rigid object to the time interval Δ**t** during which the displacement occurs.

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}$$

The **instantaneous angular speed** ω is defined as the limit of the average angular speed as Δt approaches zero. i.e.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{d\,t}$$

Angular speed has units of radians per second (rad/s).

Its dimensional formula is $M^0 L^0 T^{-1}$

We take ω to be **positive** when θ is increasing (**counterclockwise motion**) and **negative** when θ is decreasing (**clockwise motion).**

As the infinitesimal rotations follow the laws of vector addition and therefore are vector quantities.

Therefore instantaneous angular velocity is a vector quantity having direction same as angular displacement.

The direction of angular velocity can be specified in terms of x, y and z axes with the **right hand rule**.

According to it when the fingers of the right hand curl in the sense of rotation, the thumb indicates the direction of the angular displacement and angular velocity also.

Many daily life examples can help you understand the direction of angular velocity, working of a screwdriver, ceiling fans rotate clockwise to throw air downwards, potter rotates the wheel anticlockwise for the clay to move in the upward direction

7. ANGULAR ACCELERATION

If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration α_{avg}** (Greek letter alpha) of a rotating rigid object is defined as the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs:

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

The instantaneous angular acceleration is defined as the limit of the average angular acceleration as Δt approaches zero. i.e.

$$\boldsymbol{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{d\,t}}$$

Angular acceleration has units of radians per second squared (rad/s^2).

The angular acceleration α has direction same as angular velocity when a rigid object is speeding up.

Important to note

the direction of angular velocity may be imagined to be given like towards +ve x direction or –ve x direction.

The acceleration is positive if the velocity is increasing in magnitude and negative if its magnitude is decreasing

Likewise we can consider the magnitude of angular velocity increasing or decreasing.

8. RELATION BETWEEN LINEAR VELOCITY AND ANGULAR VELOCITY

Let's try to find how linear velocity of the particle is related to the angular velocity.

The reason we are looking for this is that **individual particles move with different speeds during rotation.**

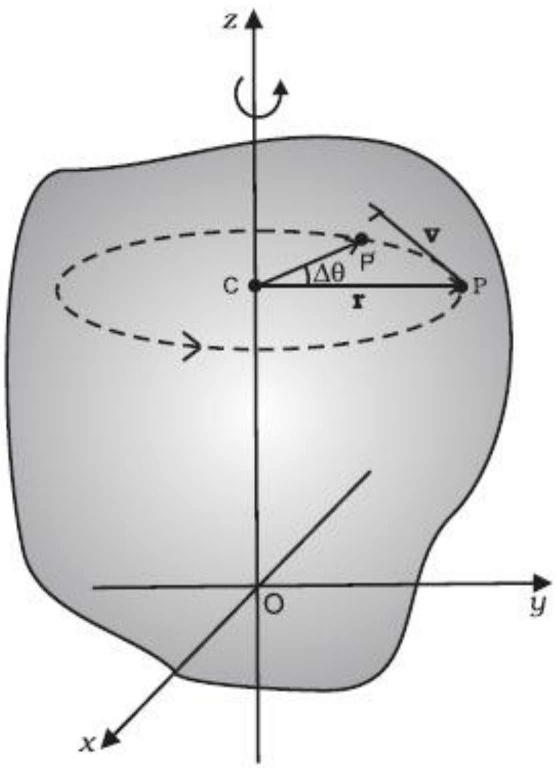
The speed depends upon the distance of the said particle from the axis of rotation.

As we know in **rotational motion of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.**

In following figure, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the z-axis).

The particle describes a circle with a centre C on the axis. The radius of the circle is r, the perpendicular distance of the point P from the axis. We also show the linear velocity vector v of the particle at P. It is along the tangent at P to the circle.

Let P' be the position of the particle after an interval of time Δt.



In the figure, showing a typical particle (at a point P) of the rigid body rotating about a fixed axis (taken as the z-axis)

The average angular speed is defined as

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}$$

while instantaneous angular speed ω is defined as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The angular displacement dθ in the interval Δt → 0 can be related to the displacement dr as

$$d\theta = \frac{dr}{r}$$

Therefore $\omega = \frac{1}{r}\frac{dr}{dt} = \frac{v}{r}$

Hence speed of the particle moving in the circle can be related to the angular speed as

$$\mathbf{v} = \omega\, \mathbf{r} \ldots\ldots\ldots\ldots\ (1)$$

Note: we are saying speed and velocity in rotation the velocity refers to the speed of motion because the direction of instantaneous velocity vector is continuously changing

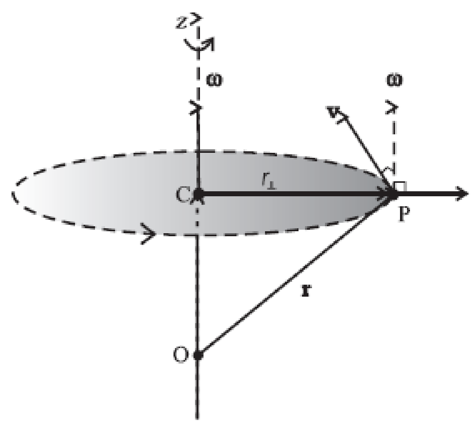
We observe that at any given instant the above relation (1) applies to all particles of the rigid body. **Particles at different distances from the axis of rotation will have different speeds but each particle will have same angular speed**. We therefore, refer to ω as the angular velocity of the whole body.

We have characterized pure translation of a body by all parts of the body having the same velocity at any instant of time. Similarly, we may characterize pure rotation by all parts of the body having the same angular velocity at any instant of time.

As for rotation about a fixed axis, the angular velocity vector lies along the axis of rotation, and points out in the direction given by right hand rule. We shall now look at what the vector product of angular velocity of the particle P, $\vec{\omega}$ and its position vector $\vec{r}$ i.e. $\vec{\omega} \times \vec{r}$

The figure shows the $\vec{\omega}$ directed along the fixed Z axis and also the position vector $\vec{r} = \overrightarrow{OP}$ of the particle at P of the rigid body with respect to the origin O.

Note that the origin is chosen to be on the axis of rotation



Now $\vec{\omega} \times \vec{r} = \vec{\omega} \times \overrightarrow{OP} = \vec{\omega} \times (\overrightarrow{OC} + \overrightarrow{CP})$

But $\vec{\omega} \times \overrightarrow{OC} = 0$ as $\vec{\omega}$ is along $\overrightarrow{OC}$

Hence $\vec{\omega} \times \vec{r} = \vec{\omega} \times \overrightarrow{CP}$

The $\vec{\omega} \times \overrightarrow{CP}$ is perpendicular to $\vec{\omega}$, i.e. to the z-axis and also to $\overrightarrow{CP}$, the radius of the circle described by the particle at P.

It is therefore, along the tangent to the circle at P. Also, the magnitude of $\vec{\omega} \times \overrightarrow{CP}$ is

$|\vec{\omega} \times \overrightarrow{CP}| = \omega\,(CP)$

Since ω and CP are perpendicular to each other.

We shall denote CP by $r_\perp$ and not by r, as we did earlier.

Thus

$|\vec{\omega} \times \vec{r}| = \omega\, r_\perp$

Here $r_\perp$ is perpendicular distance of particle P from the axis of rotation.

Therefore, we can now say $\vec{\omega} \times \vec{r}$ is a vector of magnitude $\omega\, r_\perp$ and is along the tangent to the circle described by the particle at P. The linear velocity vector $\vec{v}$ at P has the same magnitude and direction. Thus,

$\vec{v} = \vec{\omega} \times \vec{r}$ ………….. (2)

In fact, the relation, equation (2), holds good even for rotation of a rigid body with one point fixed, such as the rotation of the top where even the axis of rotation keeps changing its direction.

In this case $\vec{r}$ represents the position vector of the particle with respect to the fixed point taken as the origin.

We note that for rotation about a fixed axis, the direction of the angular velocity does not change with time.

Its magnitude may, however, change from instant to instant.

For the more general rotation, both the magnitude and the direction of angular velocity may change from instant to instant.

9. VECTOR RELATION BETWEEN LINEAR ACCELERATION AND ANGULAR ACCELERATION

For a particle P of a body, position vector $\vec{r}$ rotating about Z axis with angular velocity $\vec{\omega}$ the velocity of the particle $\vec{v}$ can be given by the equation (2) as

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Differentiating the above can give the acceleration of the particle 'P' at any instant of time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

As angular acceleration is given as $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ and linear velocity $\vec{v} = \frac{d\vec{r}}{dt}$

Hence

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Equation (3) suggests acceleration of the particle has two components given as

(i) The component represented by $\vec{a_t} = \vec{\alpha} \times \vec{r}$ is also called tangential acceleration.

As angular acceleration $\vec{\alpha}$ is directed along the axis of rotation, like angular velocity, it can be shown, that the direction of tangential is along the tangent to circular path of the particle P. It is due to the angular acceleration of the body. The magnitude of the tangential acceleration can be written as

$$|\vec{a_t}| = |\vec{\alpha} \times \vec{r}\,| = \alpha\, r_{\perp}$$

(ii) The component represented by $\vec{a_r} = \vec{\omega} \times \vec{v}$ is also called radial or centripetal acceleration. It is due to the change in direction of the motion of the particle P. This has been done in detail in the uniform circular motion. The direction of this acceleration can be determined using cross product, right hand palm rule. Using it one can conclude its direction is towards the centre of the circular path taken by the particle P, hence the name centripetal or radial. The magnitude of the centripetal acceleration can be written as

$|\vec{a_r}| = |\vec{\omega} \times \vec{v}| = \omega\, v$ (As both $\vec{\omega}$ and $\vec{v}$ are perpendicular to each other)

As linear speed of the particle is also given as

$v = \omega\, r_\perp$

Hence

$$|\vec{a_r}| = \omega\, v = \omega^2 r_\perp = \frac{v^2}{r_\perp}$$

The net acceleration of the particle is the vector sum of the two accelerations simultaneously possessed by the particle namely, tangential and centripetal. These two accelerations are perpendicular to each other. Therefore net acceleration can be written as

$$\vec{a} = \vec{a_t} + \vec{a_r}$$

$$\boldsymbol{a} = \sqrt{\mathbf{a_t}^2 + \mathbf{a_r}^2}$$

The equation (3) gives the acceleration of the particle P undergoing non-uniform circular motion.

10. EQUATIONS OF ROTATIONAL KINEMATICS FOR A RIGID BODY ROTATING WITH UNIFORMLY ANGULAR ACCELERATION

Consider a rigid body rotating with uniform angular acceleration α about a fixed axis. Let its initial angular velocity be $\boldsymbol{\omega_i}$ at time instant, $t = 0$. One can determine the angular velocity $\boldsymbol{\omega_f}$ of the body at time instant t, as

$$\alpha = \frac{d\omega}{dt}$$

$$d\omega = \alpha\, dt$$

Integrating both sides we get $\int_{\omega i}^{\omega f} d\omega = \int_0^t \alpha \, dt$

$$\omega_f - \omega i = \alpha (t - 0)$$

$$\boldsymbol{\omega_f = \omega i + \alpha t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The above can help us in finding angular velocity at any time if we know initial angular velocity of a body rotating uniform angular acceleration.

One can also determine the angular displacement of the body, discussed above, in a given time interval.

As

$$\omega = \frac{d\theta}{dt}$$

$$d\theta = \omega \, dt$$

Integrating above equation with limits of angular position at time instant t = 0 be θ_i and at time instant t the angular position is θ_f

$$\theta_f - \theta_i = \int_0^t (\omega_i + \alpha t) \, dt \qquad \text{(Using equation (4))}$$

$$\theta_f - \theta_i = \left(\omega_i \, t + \frac{1}{2} \alpha \, t^2\right)_0^t$$

Angular displacement $\boldsymbol{\Delta\theta} = \theta_f - \theta_i = \boldsymbol{\omega_i t + \frac{1}{2} \alpha \, t^2}$ ………………… (5)

Using angular displacement, no of revolution made by the rotating body in the given interval as

$$\text{No of revolutions, } n = \frac{\text{angular displacement}}{2\pi} = \frac{\theta_f - \theta_i}{2\pi}$$

Also one can determine the relation between the initial and final angular velocities of a body with its angular acceleration and angular displacement.

As

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{dt} \times \frac{d\theta}{d\theta} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}$$

$$\alpha = \omega \frac{d\omega}{d\theta}$$

Integrating above equation with limits of angular velocity of body as ω_i when its angular position is θ_i and angular velocity as ω_f , when its angular position is θ_f.

$$\alpha \, d\theta = \omega \, d\omega$$

$$\int_{\theta i}^{\theta f} \alpha \, d\theta = \int_{\omega i}^{\omega f} \omega \, d\omega$$

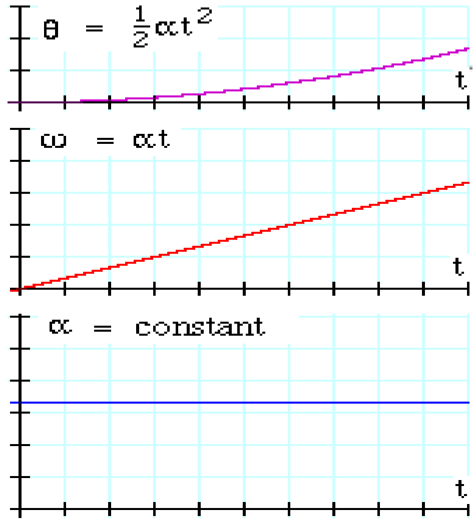
$$\alpha \, (\theta_f - \theta_i) = \left(\frac{\omega^2}{2} \right)_{\omega i}^{\omega f}$$

$$\alpha \, (\theta_f - \theta_i) = \frac{\omega_f^2 - \omega_i^2}{2}$$

$$\omega_f^2 - \omega_i^2 = 2 \, \alpha \, (\theta_f - \theta_i)$$

$$\boldsymbol{\omega_f^2 - \omega_i^2 = 2 \, \alpha \, \Delta\theta} \qquad \textbf{……………………….(6)}$$

The variation of angular displacement, angular velocity and angular acceleration with time for a body rotating with uniform angular acceleration are shown below



EXAMPLE

The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform?

(ii) How many revolutions does the engine make during this time?

SOLUTION

(i) We shall use $\omega_f = \omega_i + \alpha t$

ω_i = initial angular speed in rad/s = $2\pi \times$ angular speed in rev/s

$$\omega_i = 2\pi \times \frac{\text{angular speed in rev/min}}{60}$$

$$\omega_i = 2\pi \times \frac{1200}{60} \text{rad/s} = 40\ \pi \text{ rad /s}$$

Similarly, ω_f = final angular speed in rad/s

$$\omega_f = 2\pi \times \frac{3120}{60} \text{ rad /s}$$

$\omega_f = 104\ \pi$ rad/s

Angular acceleration $\alpha = \frac{\omega_f - \omega_i}{t}$

$\alpha = \frac{104\,\pi - 40\,\pi}{16}$

$\alpha = 4\,\pi$ rad /s^2

(ii) The angular displacement in time t is given by

$\Delta\boldsymbol{\theta} = \theta_f - \theta i = \omega i\, t + \frac{1}{2}\alpha\, t^2$

$\Delta\boldsymbol{\theta} = 40\,\pi \times 16 + \frac{1}{2} \times 4\,\pi \times 16^2$ rad

$\Delta\boldsymbol{\theta} = 640\,\pi + 512\,\pi$ rad

$\Delta\boldsymbol{\theta} = 1152\,\pi$ rad

Number of revolutions $n = \frac{\text{angular displacement}}{2\pi}$

$n = \frac{1152\,\pi}{2\pi} = 576$

EXAMPLE

Suppose a train accelerates from rest, giving its 0.350 m radius wheels an angular acceleration of 0.250 rad/s^2. After the wheels have made 200 revolutions (assume no slippage):

(a) How far has the train moved down the track?

(b) What are the final angular velocity of the wheels and the linear velocity of the train?

SOLUTION

Angular acceleration of the wheels α =0.250 rad/s^2

Radius of wheel R = 0.350 m

No. of revolutions n = 200

(a) As Number of revolutions $n = \frac{\text{angular displacement}}{2\pi}$

Therefore, angular displacement $= n \times 2\pi = 200 \times 2\pi = 400\pi$ rad

Linear displacement of the wheel $\Delta x = R \times \Delta\theta$

$\Delta x = 0.350 \times 400\pi = 140\pi = 439.6$ m

(b) As $\omega_f^2 - \omega_i^2 = 2\,\alpha\Delta\theta$

ωi = initial angular speed= 0

(c) $\omega_f^2 = 2\,2\,\alpha\Delta\theta + \omega_i^2 = 2 \times 0.250 \times 400\pi + 0$

$\omega_f = \sqrt{2 \times 0.250 \times 400\pi} = \sqrt{628} = 25.06$ rad/s

As linear speed of the wheel or train $v = \omega R$

Therefore $v = 25.06 \times 0.350 = \frac{8.77\text{m}}{\text{s}}$

11. SUMMARY

- When rigid object moves from position about any axis there is angular displacement θ
- If it displaces from θ_i to position θ_f in a time interval Δt the reference line fixed to the object sweeps out an angle $\Delta\theta = \theta_f - \theta_i$ defined as the **angular displacement** of the rigid object.
- Its **direction** is given with the help of **right hand rule**. According to it when the fingers of the right hand curl in the sense of rotation, the thumb indicates the direction of the angular displacement.
- The average **angular speed $\boldsymbol{\omega_{avg}}$** as the ratio of the angular displacement of a rigid object to the time interval Δ**t** during which the displacement occurs.

$$\boldsymbol{\omega_{avg}} = \frac{\Delta\boldsymbol{\theta}}{\Delta t} = \frac{\boldsymbol{\theta_f - \theta_i}}{\Delta t}$$

- The **instantaneous angular speed** ω is defined as the limit of the average angular speed as Δt approaches zero. i.e.

$$\boldsymbol{\omega} = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- The **average angular acceleration** $\boldsymbol{\alpha_{avg}}$ of a rotating rigid object is defined as the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs:

$$\boldsymbol{\alpha_{avg}} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}.$$

- The **instantaneous angular acceleration** is defined as the limit of the average angular acceleration as Δt approaches zero. i.e.

$$\boldsymbol{\alpha} = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- The linear velocity of a particle of a rigid body rotating with angular velocity $\vec{\omega}$ about a fixed axis is given by, $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector of the particle with respect to an origin along the fixed axis. The relation applies even to more general rotation of a rigid body with one point fixed. In that case r is the position vector of the particle with respect to the fixed point taken as the origin.

- The net acceleration of the particle is the vector sum of the two accelerations simultaneously possessed by the particle namely**, tangential and centripetal**. These two **accelerations are perpendicular to each other**. Therefore net acceleration can be written as

$$\vec{a} = \vec{a_t} + \vec{a_r} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

Where tangential acceleration $|\vec{a_t}| = \alpha\, r_\perp$ and

Centripetal acceleration $|\vec{a_r}| = \omega\, v = \omega^2 r_\perp = \frac{v^2}{r_\perp}$

Magnitude of net acceleration $\boldsymbol{\alpha = \sqrt{\alpha_t{}^2 + \alpha_r{}^2}}$

- When a rigid object is rotating about a fixed axis, every particle on the object rotates through the same angle in a given time interval and has the **same angular speed and the same angular acceleration.** Therefore, like the angular position θ, the quantities ω and α characterize the rotational motion of the entire rigid object as well as individual particles in the object.

- For a body rotating about an axis with constant angular acceleration relation between quantities can be written as. Let $\boldsymbol{\omega_i}$ be initial angular veloity, $\boldsymbol{\omega_f}$ be final angular velocity and $\boldsymbol{\alpha}$ be angular acceleration.

- The equations of motion for rotation about a fixed axis with constant angular acceleration

  Then one can write

  $$\omega_f = \omega i + \alpha t$$

  $$\Delta\theta = \theta_f - \theta i = \omega i\, t + \frac{1}{2}\alpha\, t^2$$

  $$\omega_f^2 - \omega_i^2 = 2\, \alpha\, \Delta\theta$$

- These equations are similar to equations of motion in one dimension

  v = u + at or $v(t) = v(0) + at$

  S= u t + $\frac{1}{2}at^2$ **or** $x(t-0) = x(0) + v(0)t + \frac{1}{2}a(t-0)^2$

  $v^2 - u^2 = 2as$ **or** $v(t)^2 - v(0)^2 = 2ax(t-0)$